

HOLOMORPHIC EXTENDIBILITY AND THE ARGUMENT PRINCIPLE

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Dedicated to the memory of Herb Alexander

ABSTRACT The paper gives the following characterization of the disc algebra in terms of the argument principle: A continuous function f on the unit circle $b\Delta$ extends holomorphically through the unit disc if and only if for each polynomial P such that $f + P \neq 0$ on $b\Delta$ the change of argument of $f + P$ around $b\Delta$ is nonnegative.

1. Introduction and the main result

The present paper deals with the problem of characterizing the continuous functions on the unit circle which extend holomorphically into the unit disc, in terms of the argument principle. Generalizing some results of H.Alexander and J.Wermer [AW], E.L.Stout [S] obtained a characterization of continuous functions on boundaries of certain domains D in \mathbb{C}^n , $n \geq 1$ which extend holomorphically through D , in terms of a generalized argument principle. In the special case of Δ , the open unit disc in \mathbb{C} , a version of his result is

THEOREM 1.0 [S] *A continuous function f on $b\Delta$ extends holomorphically through Δ if and only if*

$$\left. \begin{array}{l} \text{for each polynomial } Q \text{ of two complex variables such} \\ \text{that } Q(z, f(z)) \neq 0 \text{ (} z \in b\Delta \text{) the change of argument of} \\ \text{the function } z \mapsto Q(z, f(z)) \text{ around } b\Delta \text{ is nonnegative.} \end{array} \right\} \quad (1.1)$$

J.Wermer [W] showed that it suffices to assume (1.1) only for polynomials of the form $Q(z, w) = w + P(z)$ provided that f is smooth and asked whether the same holds for continuous functions. In the present paper we prove that this indeed is the case:

THEOREM 1.1 *A continuous function f on $b\Delta$ extends holomorphically through Δ if and only if*

$$\left. \begin{array}{l} \text{for each polynomial } P \text{ such that } f + P \neq 0 \text{ on } b\Delta \text{ the} \\ \text{change of argument of } f + P \text{ around } b\Delta \text{ is nonnegative.} \end{array} \right\} \quad (1.2)$$

Note that the only if part is an obvious consequence of the argument principle. In fact, if f admits a holomorphic extension \tilde{f} then the change of argument of $f + P$ around $b\Delta$ equals 2π times the number of zeros of $\tilde{f} + P$ in Δ .

2. The Morera condition

LEMMA 2.1 *Let f be a continuous function on $b\Delta$ which satisfies (1.2). Then*

$$\int_{b\Delta} f(z)dz = 0.$$

Proof. Suppose that $\int_{b\Delta} f(z)dz \neq 0$. With no loss of generality assume that

$$\frac{1}{2\pi i} \int_{b\Delta} f(z)dz = 1.$$

Then $z \mapsto zf(z) - 1$ is a continuous function on $b\Delta$ with zero average. Since the polynomials in z and \bar{z} are dense in $C(b\Delta)$ it follows that there are polynomials P and Q and a continuous function g on $b\Delta$ such that

$$|g(z)| \leq 1/2 \quad (z \in b\Delta) \tag{2.1}$$

and such that

$$zf(z) - 1 = zP(z) + \overline{zQ(z)} + g(z) \quad (z \in b\Delta).$$

It follows that

$$z[f(z) - P(z) - Q(z)] \in 1 + g(z) + i\mathbb{R} \quad (z \in b\Delta)$$

which, by (2.1), implies that

$$\frac{1}{2} \leq \Im(z[f(z) - P(z) - Q(z)]) \leq \frac{3}{2} \quad (z \in b\Delta),$$

so the change of argument of $z \mapsto z[f(z) - P(z) - Q(z)]$ around $b\Delta$ equals zero. Thus, $f - P - Q \neq 0$ on $b\Delta$ and the change of argument of $f - P - Q$ around $b\Delta$ equals -2π , contradicting the assumption that f satisfies (1.2). This completes the proof.

Before proceeding to the proof of Theorem 1.1 observe that Lemma 2.1 provides a simple alternative proof of Theorem 1.0; in fact, it provides a proof of the following corollary which sharpens Theorem 1.0.

COROLLARY 2.1 *A continuous function f on $b\Delta$ extends holomorphically through Δ if and only if for each nonnegative integer n and each polynomial P such that $z^n f(z) + P(z) \neq 0$ ($z \in b\Delta$) the change of argument of the function $z \mapsto z^n f(z) + P(z)$ around $b\Delta$ is nonnegative.*

Proof. If f satisfies the condition then Lemma 2.1 implies that

$$\int_{b\Delta} z^n f(z)dz = 0$$

for each nonnegative integer n . It is well known that this implies that f extends holomorphically through Δ . This completes the proof.

3. Proof of Theorem 1.1

LEMMA 3.1 *Suppose that f is a continuous function on $b\Delta$ that satisfies (1.2). Then for each $w \in \mathbb{C} \setminus \overline{\Delta}$ the function $z \mapsto f(z)/(z - w)$ satisfies (1.2).*

Proof. Suppose that $w \in \mathbb{C} \setminus \overline{\Delta}$ and assume that P is a polynomial such that

$$\frac{f(z)}{z - w} + P(z) \neq 0 \quad (z \in b\Delta).$$

Then

$$f(z) + (z - w)P(z) \neq 0 \quad (z \in b\Delta)$$

and since f satisfies (1.2) it follows that the change of argument of the function $z \mapsto f(z) + (z - w)P(z)$ around $b\Delta$ is nonnegative. Since

$$\frac{f(z)}{z - w} + P(z) = \frac{1}{z - w} [f(z) + (z - w)P(z)] \quad (z \in b\Delta)$$

and since the change of argument of $z \mapsto 1/(z - w)$ around $b\Delta$ is zero it follows that the change of argument of

$$z \mapsto \frac{f(z)}{z - w} + P(z) \tag{3.1}$$

around $b\Delta$ is nonnegative. This completes the proof.

Proof of Theorem 1.1. Suppose that f satisfies (1.2). By Lemma 3.1, for each $w \in \mathbb{C} \setminus \overline{\Delta}$ the function (3.1) satisfies (1.2). By Lemma 2.1 it follows that

$$\int_{b\Delta} \frac{f(z)}{z - w} dz = 0 \quad (w \in \mathbb{C} \setminus \overline{\Delta}). \tag{3.2}$$

It is well known that (3.2) implies that f extends holomorphically through Δ . This completes the proof.

4. An example

Theorem 1.1 gives a simple characterization of continuous functions on $b\Delta$ that extend holomorphically to Δ in terms of the argument principle. One can ask whether one can further simplify this characterization. J.Wermer [W] showed that in Theorem 1.1 it is not enough to assume (1.2) for polynomials of the form $P(z) = az + b$, $a, b \in \mathbb{C}$. In this section we sharpen this by showing that for Theorem 1.1 to hold (1.2) must hold for polynomials of arbitrarily large degree.

PROPOSITION 4.1 *For every $n_0 \in \mathbb{N}$ there is a continuous function f on $b\Delta$ such that whenever P is a polynomial of degree not exceeding n_0 such that $f + P \neq 0$ on $b\Delta$, then the change of argument of $f + P$ around $b\Delta$ is nonnegative, yet f does not extend holomorphically through Δ .*

Proof. Let $n_0 \in \mathbb{N}$ and let $n \in \mathbb{N}$, $n \geq n_0 + 1$. Set

$$f(z) = z^n + \frac{1}{2z} \quad (z \in b\Delta).$$

We show that for every polynomial P of degree not exceeding n_0 such that $f + P \neq 0$ on $b\Delta$, the change of argument of $f + P$ around $b\Delta$ is nonnegative. Assume, contrarily to what we want to prove, that there is a polynomial P , $\deg(P) \leq n_0$, such that $f + P$ has no zero on $b\Delta$, and such that the change of argument of $f + P$ around $b\Delta$ is negative.

Since $f + P$ is rational with a single pole in Δ the argument principle implies that the change of argument of $f + P$ around $b\Delta$ equals $2\pi(\nu - 1)$ where ν is the number of zeros of $f + P$ on Δ . By our assumption, $2\pi(\nu - 1) < 0$, so $f + P$ has no zero on Δ . The zeros of $f + P$ are the zeros of

$$z^{n+1} + zP(z) + \frac{1}{2}. \quad (4.1)$$

Since $n \geq n_0 + 1$ and since the degree of P does not exceed n_0 it follows that the leading term in (4.1) is z^{n+1} . Since the constant term in (4.1) is $1/2$ it follows that the product of zeros of (4.1) equals $1/2$ which implies that at least one of the zeros of (4.1) is contained in Δ , so at least one of the zeros of $f + P$ is contained in Δ , a contradiction. This completes the proof.

5. Holomorphic extendibility to finite Riemann surfaces

Theorem 1.0 has yet another, less elementary but even shorter proof using Wermer's maximality theorem: Suppose that f is a continuous function on $b\Delta$ which satisfies (1.1) and which does not extend holomorphically through Δ . By Wermer's maximality theorem [H] the polynomials in z and f are dense in $C(b\Delta)$. In particular, there is a polynomial Q such that

$$|Q(z, f(z)) - \bar{z}| < \frac{1}{2} \quad (z \in b\Delta).$$

Obviously $Q(z, f(z)) \neq 0$ ($z \in b\Delta$) and the change of argument of $z \mapsto Q(z, f(z))$ around $b\Delta$ equals the change of argument of $z \mapsto \bar{z}$ around $b\Delta$ which equals -2π . Thus, the change of argument of $z \mapsto Q(z, f(z))$ around $b\Delta$ is negative which contradicts the fact that f satisfies (1.1) and so completes the proof of Theorem 1.1. This proof of Stout's theorem was the first that the author found. Only after a careful reading of Cohen's proof of Wermer's maximality theorem [C] the author found the proof of Lemma 2.1 which gives a more elementary proof of Stout's theorem. J. Wermer has kindly pointed out to the author that the preceding proof generalizes to finite Riemann surfaces which gives

THEOREM 5.1 *Let X be a finite Riemann surface with boundary Γ . Let A be the algebra of all continuous functions on Γ which extend holomorphically through X . A continuous function f on Γ extends holomorphically through X if and only if for every polynomial P with coefficients in A such that $P(f) \neq 0$ on Γ the change of argument of $P(f)$ along Γ is nonnegative.*

Proof. Suppose that $f \in C(\Gamma)$ has the property that the change of argument of $P(f)$ along Γ is nonnegative whenever P is a polynomial with coefficients in A such that $P(f) \neq 0$ on

Γ . Suppose that f does not extend holomorphically through X . By the maximality of A in $C(\Gamma)$ [R] the functions of the form $P(f)$ where P is a polynomial with coefficients in A , are dense in $C(\Gamma)$. Choose $g \in C(\Gamma)$ such that $|g| = 1$ on Γ and such that the change of argument of g along Γ equals -2π . There is a polynomial P with coefficients in A such that $|P(f)(z) - g(z)| < 1/2$ ($z \in \Gamma$). Since $|g| = 1$ on Γ the change of argument of $P(f)$ along Γ equals the change of argument of g along Γ . So the change of argument of $P(f)$ along Γ is negative, contradicting the hypothesis. This proves that f extends holomorphically through X . The only if part follows from the argument principle. This completes the proof.

Mark Agranovsky observed that a substantially longer argument in the proof of Theorem 1.1 in the original version of the paper can be replaced by Lemma 3.1 which is due to him. The author is grateful to him for his kind permission to include Lemma 3.1 into the final version of the paper. The author is also grateful to John Wermer for pointing out that Theorem 5.1 follows from the maximality theorem.

The author is grateful to Larry Zalcman for the kind invitation to include this paper into the proceedings of the Nahariya conference.

This work was supported in part by the Ministry of Education, Science and Sport of Slovenia through research program Analysis and Geometry, Contract No. P1-0291.

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